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Some Applications of Kober Operator of the Second Kind and Integral Transforms in Statistical Distributions

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Abstract

In this paper, we studied Kober integral operator of the second kind of order α, and we found Mellin transform of this operator, we discussed the relation between Mellin transform of Kober operator and the expected value of product of two random variables one of them has a type-1 beta density, and we also discussed a pathway Kober operator of the second kind as a generalization of some densities

Keywords: Kober Operator, Mellin Transform, Beta Density, Expected Value

Introduction

In 1940 Kober introduced some generalization of the basic Riemann-Liouville fractional integral and differential operators see [1]. The Kober fractional integral operator related to some statistical distributions. This relation will also allow us to come up with a general definition for fractional integral Kober operator of the second kind see [6]. One of the generalization considered is the pathway idea where one can move from one family of operators to another family and yet another family and eventually end up with an exponential form see [5].

1. Mellin Transform of The Kober Operator of The Second Kind

The Kober fractional integral of the second of order α is defined and denoted as

$$K_{2,x,\beta}^{-\alpha} f = \frac{x^{\beta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1}\, t^{-\beta-\alpha} f(t)\, dt \qquad (2.1)$$

Mellin transform of the function $f(x)$ is defined and denoted as[]

$$f^{*}(s) = M\{f(x)\} = \int_{0}^{\infty} x^{s-1} f(x)\, dx \qquad (2.2)$$

Then the Mellin transform of the Kober operator of the second kind is given by

$$M\{K_{2,x,\beta}^{-\alpha} f\} = \int_{0}^{\infty} x^{s-1} \left[\frac{x^{\beta}}{\Gamma(\alpha)} \int_{x}^{\infty} (t-x)^{\alpha-1}\, t^{-\beta-\alpha} f(t)\, dt\right] dx \qquad (2.3)$$

Interchanging the integrals, we get

$$M\{K_{2,x,\beta}^{-\alpha} f\} = \int_{t=0}^{\infty} t^{-\beta-\alpha} f(t) \left[\int_{x=0}^{t} \frac{x^{\beta+s-1}(t-x)^{\alpha-1}}{\Gamma(\alpha)}\, dx\right] dt$$

$$= \int_{t=0}^{\infty} t^{-\beta-\alpha} f(t) \left[\int_{x=0}^{t} \frac{x^{\beta+s-1}\left(1-\frac{x}{t}\right)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)}\, dx\right] dt \qquad (2.4)$$

by using the following substitution

$$y = \frac{x}{t} \quad \text{or} \quad x = ty \quad \text{and} \quad dx = t\, dy$$

$$\text{and } y = 0 \text{ when } x = 0, \quad y = 1 \text{ when } x = t$$

we get

$$M\{K_{2,x,\beta}^{-\alpha} f\} = \int_{t=0}^{\infty} t^{-\beta-\alpha} f(t)\left[\int_{y=0}^{1} \frac{(ty)^{\beta+s-1}(1-y)^{\alpha-1}\, t^{\alpha}}{\Gamma(\alpha)}\, dy\right] dt$$

$$= \frac{1}{\Gamma(\alpha)} \int_{t=0}^{\infty} t^{s-1} f(t)\, dt \int_{y=0}^{1} y^{\beta+s-1}(1-y)^{\alpha-1}\, dy \qquad (2.5)$$

in equation(2.5), first integration is equal to $f^*(s)$ and second integration is equal to $beta(\beta+s,\alpha) = \frac{\Gamma(\alpha)\,\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)}$ where $beta(n,m) = \int_{y=0}^{1} y^{n-1}(1-y)^{m-1}\, dy$, then (2.5) becomes

$$M\{K_{2,x,\beta}^{-\alpha} f\} = \frac{\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)} f^*(s) \qquad (2.6)$$

2. The Relation between Kober Operator of the Second Kind and Type-1 Beta density:

As we know that type-1 beta density with the parameters n , m is defined as

$$\frac{\Gamma(n+m)}{\Gamma(n)\,\Gamma(m)} \int_{y=0}^{1} y^{n-1}(1-y)^{m-1}\, dy \qquad (3.1)$$

If we have two independently distributed random variables x_1 , x_2 , and x_1 has a type-1 beta density with the parameters $\beta+1$, α denoted by $f_1(x_1)$, and x_2 has an arbitrary density $f_2(x_2)$, then the density of product $u = x_1 x_2$ can be written as

$$g(u) = \int_{v} \frac{1}{v} f_1\left(\frac{u}{v}\right) f_2(v)\, dv \qquad (3.2)$$

where $v = x_2$, then $x_1 = \frac{u}{x_2} = \frac{u}{v}$ and $\frac{1}{v}$ is the Jacobian

Taking f_1 as type-1 beta with the parameters $\beta+1$, α, $v \to \infty$ when $x_1 = 0$, $v = u$ when $x_1 = 1$, we get

$$g(u) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\,\Gamma(\alpha)} \int_{v=u}^{\infty} \frac{1}{v} \left(\frac{u}{v}\right)^{\beta} \left(1-\frac{u}{v}\right)^{\alpha-1} f_2(v)\, dv$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{u^{\beta}}{\Gamma(\alpha)} \int_{v=u}^{\infty} v^{-\beta-\alpha} (v-u)^{\alpha-1} f_2(v)\, dv$$

$$= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} K_{2,x,\beta}^{-\alpha} f_2 \qquad (3.3)$$

that means

$$K_{2,x,\beta}^{-\alpha} f_2 = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} g(u) \qquad (3.4)$$

We can find Mellin transform of both sides of(3.4), we get

$$M\{K_{2,x,\beta}^{-\alpha} f_2\} = M\left\{\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} g(u)\right\} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} g^*(s) \qquad (3.5)$$

The expected value of u^{s-1} is equal to

$$E(x_1 x_2)^{s-1} = E({x_1}^{s-1}\, {x_2}^{s-1}) \qquad (3.6)$$

since x_1 and x_2 are independently distributed random variables[9], then

$$E(x_1 x_2)^{s-1} = E({x_1}^{s-1})\, E({x_2}^{s-1}) \qquad (3.7)$$

in equation(3.7), first expected value

$$E({x_1}^{s-1}) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)} \qquad (3.8)$$

and second expected value is the Mellin transform of the density function f_2 , then

$$E(u^{s-1}) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)} f_2^*(s) \qquad (3.9)$$

We can find Mellin transform of both sides of (3.4), we get

$$M\{K_{2,x,\beta}^{-\alpha} f_2\} = M\left\{\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, g(u)\right\}$$
$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, g^*(s) \qquad (3.10)$$

since $E(u^{s-1})$ is the Mellin transform of $g(u)$, then

$$M\{K_{2,x,\beta}^{-\alpha} f_2\} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, g^*(s)$$
$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, E(u^{s-1}) \qquad (3.11)$$

From equation(3.8)

$$M\{K_{2,x,\beta}^{-\alpha} f_2\}$$
$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\,\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\,\frac{\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)}\, f_2^*(s)$$
$$= \frac{\Gamma(\beta+s)}{\Gamma(\alpha+\beta+s)}\, f_2^*(s) \qquad (3.12)$$

equation (3.12) show the relation between Kober operator of the second kind and type-1 beta density by using Mellin transform.

3. A Pathway Kober Operator of The Second Kind:

We have two types of pathway Kober operator of the second kind densities :

(i) Left pathway density :

$$f_1(x_1) = c_1\, {x_1}^{\gamma}\left[1 - a(1-p){x_1}^{\delta}\right]^{\frac{\mu}{1-p}} \quad for \quad p < 1 \qquad (4.1)$$

where

$$c_1 = \frac{\delta\,[a(1-p)]^{\frac{\gamma+1}{\delta}}\,\Gamma(\frac{\gamma+1}{\delta} + \frac{\mu}{1-p} + 1)}{\Gamma(\frac{\gamma+1}{\delta})\Gamma(\frac{\mu}{1-p}+1)} \quad for \ \frac{\gamma+1}{\delta} > 0\,, \quad \frac{\mu}{1-p} > 0 \qquad (4.2)$$

From equation(3.2), if x_2 has an arbitrary density $f_2(x_2)$ and x_1 has a pathway density, then the density of $u = x_1\, x_2$ is given by

$$g(u) = c_1 \int_v \frac{1}{v}\left(\frac{u}{v}\right)^{\gamma}\left[1 - a(1-p)\left(\frac{u}{v}\right)^{\delta}\right]^{\frac{\mu}{1-p}} f_2(v)\ dv$$
$$= c_1 u^{\gamma} \int_{v=[a(1-p)]^{\frac{1}{\delta}}}^{\infty} v^{-\gamma-1}\left[1 - a(1-p)\left(\frac{u}{v}\right)^{\delta}\right]^{\frac{\mu}{1-p}} f_2(v)\ dv \qquad (4.3)$$

Where p describes a path of movement Kober operator of the second kind. In the limit when $\rightarrow {}^{(-)}1$, from definition of exponential function

$$e^{ac} = \lim_{x\to 0}(1+ax)^{\frac{c}{x}} \qquad (4.4)$$

The equation(4.3) will go to

$$\lim_{p\to {}^{(-)}1} g(u) = C_1 u^{\gamma}\int_{v=0}^{\infty} v^{-\gamma-1}\, e^{-a\,\mu\left(\frac{u}{v}\right)^{\delta}} f_2(v)\ dv \qquad (4.5)$$

where

$$C_1 = \frac{\delta\,(a\mu)^{\frac{\gamma+1}{\delta}}}{\Gamma(\frac{\gamma+1}{\delta})} \qquad (4.6)$$

(ii) Right pathway density:

If $p > 1$ then $(1-p)$ changes to $-(\,p-1\,)$ and (4.1) becomes

$$f_1(x_1) = c_2\, {x_1}^{\gamma}\left[1 + a(p-1){x_1}^{\delta}\right]^{\frac{-\mu}{p-1}} \quad for \quad p > 1 \qquad (4.7)$$

where

$$c_2 = \frac{\delta\,[a(p-1)]^{\frac{\gamma+1}{\delta}}\Gamma(\frac{\gamma}{p-1})}{\Gamma\left(\frac{\gamma+1}{\delta}\right)\Gamma\left(\frac{\mu}{p-1}-\frac{\mu+1}{\delta}\right)} \quad for\ \frac{\gamma+1}{\delta}>1\,,$$

$$\frac{\mu}{p-1} > \frac{\mu+1}{\delta} \quad (4.8)$$

the density of $u = x_1\, x_2$ is given by

$$g(u) = c_2 \int_v \frac{1}{v}\left(\frac{u}{v}\right)^{\gamma}\left[1+a(p-1)\left(\frac{u}{v}\right)^{\delta}\right]^{\frac{-\mu}{p-1}} f_2(v)\ dv$$

$$= c_2 u^{\gamma}\int_{v=[-a(p-1)]^{\frac{1}{\delta}}}^{\infty} v^{-\gamma-1}\left[1 + a(p-1)\left(\frac{u}{v}\right)^{\delta}\right]^{\frac{-\mu}{p-1}} f_2(v)\ dv \quad (4.9)$$

Similarly, we can use the definition of exponential function, we get

$$\lim_{p\to(+)1} g(u) = C_2 u^{\gamma}\int_{v=0}^{\infty} v^{-\gamma-1}\, e^{-a\,\mu\left(\frac{u}{v}\right)^{\delta}} f_2(v)\ dv \quad (4.10)$$

where

$$C_2 = \frac{\delta\,(a\mu)^{\frac{\gamma+1}{\delta}}}{\Gamma(\frac{\gamma+1}{\delta})} \quad (4.11)$$

then $C_1 = C_2 = C$, and $\lim_{p\to(-)1} g(u) = \lim_{p\to(+)1} g(u) = \lim_{p\to 1} g(u)$

We can rewriting $f_2(v)$ as a function of $v^{-\delta}$, let us consider $f_2(v) = f_3(v^{-\delta})$ by substituting $t = v^{-\delta}$ or $v = t^{\frac{-1}{\delta}}$ and $dv = \frac{-1}{\delta}\, t^{-\left(\frac{\delta+1}{\delta}\right)}\, dt$, (4.5) and (4.10) become

$$\lim_{p\to 1} g(u) = C\frac{u^{\gamma}}{\delta}\int_{t=0}^{\infty} t^{\frac{\gamma+1}{\delta}}\, e^{-a\,\mu\, u^{\delta} t}\ t^{-\left(\frac{\delta+1}{\delta}\right)} f_3(t)\ dv$$

$$= C\frac{u^{\gamma}}{\delta}\int_{t=0}^{\infty} t^{\frac{\gamma-\delta}{\delta}}\, e^{-a\,\mu\, u^{\delta} t}\ f_3(t)\ dv \quad (4.12)$$

Let us consider $s = a\,\mu\, u^{\delta} > 0$ and $m = \frac{\gamma-\delta}{\delta}$, (4.12) becomes

$$\lim_{p\to 1} g(u) = C\frac{u^{\gamma}}{\delta}\int_{t=0}^{\infty} t^{m}\, e^{-st}\ f_3(t)\ dv \quad (4.13)$$

From properties of Laplace transform , we get

$$\lim_{p\to 1} g(u) = (-1)^m \frac{d^m F_3(s)}{ds^m} \quad (4.14)$$

where $F_3(s)$ is a Laplace transform of $f_3(t)$.

If m is an integer number, then we can find $\lim_{p\to 1} g(u)$, if m is a fractional number we can solve (4.14) by using the fractional integral operators.

Conclusion:

In this paper, we present the Kober operator of the second kind as a density of product of two statistically independently distributed real positive random variables, this operator and its pathway density related to Mellin and Laplace transforms, this relation allow us to work with more fractional integral operators by using some properties of these transforms.

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